MATH 2020 Advanced Calculus II

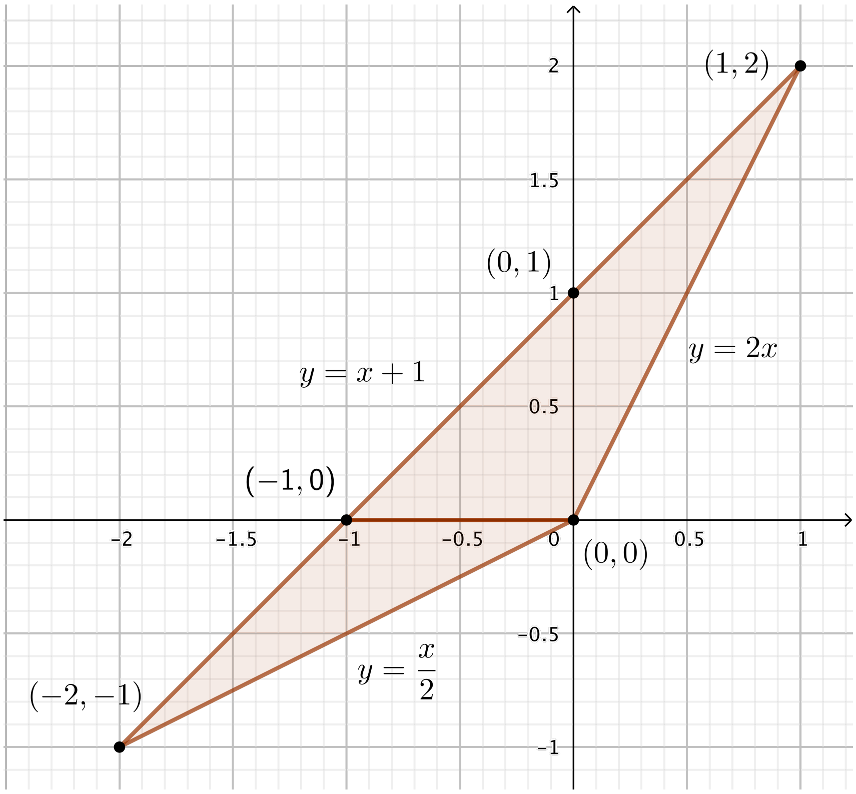
Tutorial 3

1. Sketch the region whose area can be obtained by computing the following integral

$$\int_0^2 \int_{y-1}^{\frac{y}{2}} dxdy + \int_{-1}^0 \int_{y-1}^{2y} dxdy,$$

and find the area of this region.

Solution. The region is given by



The area is then the value of the given integral:

$$\begin{aligned}
&\int_0^2 \int_{y-1}^{\frac{y}{2}} dxdy + \int_{-1}^0 \int_{y-1}^{2y} dxdy \\
=& \int_0^2 \left(1 - \frac{y}{2}\right) dy + \int_{-1}^0 (1-y)dy \\
=& \left(2 - \frac{2^2}{4}\right) + \left(1 - \frac{1}{2}\right) \\
=& \frac{3}{2}
\end{aligned}$$

2. Find the area of the region lying in the first quadrant and bounded by the curve $x^3 + y^3 = 3xy$. (The curve is called the *Folium of Descartes*.)

Solution. First we rewrite the equation of the given curve in polar form ($x = r\cos\theta, y = r\sin\theta$):

$$x^3 + y^3 = 3xy \iff r^3(\cos^3\theta + \sin^3\theta) = 3r^2\cos\theta\sin\theta$$
$$\iff r = \frac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}.$$

Notice that the set of θ for which the curve lies in the first quadrant is $\left[0, \frac{\pi}{2}\right]$.

Now the area of the region is given by

$$\begin{aligned}
\iint_R dxdy &= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta \\
&= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta\sin^2\theta}{(\cos^3\theta + \sin^3\theta)^2} d\theta \\
&= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta\sec^2\theta}{(1 + \tan^3\theta)^2} d\theta \\
&= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{3}d(\tan^3\theta)}{(1 + \tan^3\theta)^2} \\
&= \frac{3}{2}\left[-\frac{1}{1 + \tan^3\theta}\right]_0^{\frac{\pi}{2}} \\
&= \frac{3}{2}.
\end{aligned}$$

3. Compute the volume of the solid bounded by the unit sphere $x^2 + y^2 + z^2 = 1$ without using polar coordinates.

Solution. By symmetry, the volume is equal to 8 times the volume of the piece lying in the first octant:

$$\begin{aligned}
\text{volume} &= 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dzdydx \\
&= 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2} dydx \\
&= 8\int_0^1 \frac{\pi}{4}(1 - x^2)dx \qquad \left(\text{recall } \int_0^a \sqrt{a^2 - y^2}dy = \frac{\pi}{4}a^2\right) \\
&= 2\pi\left[x - \frac{x^3}{3}\right]_0^1 \\
&= \frac{4\pi}{3}
\end{aligned}$$